10- -21

Last time: Chain "Roolz" Rule, $\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial t_2} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x}{\partial t_2} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$

Implicit function theorem: Let F be a function w/ $\frac{\partial F}{\partial x} \neq 0$ and $\frac{\partial f}{\partial x_i}$ exts. then on the locus (set of points) of $F(x_1, x_2, x_3 \cdots x_n) = 0$ we have locally $x_n = f(x_1, \cdots, x_{n-1})$ and $\frac{\partial F}{\partial x_i} = -\frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_n}$

Pf (IFT Derivative Formula): Apply a partial derivative to F using chain rule:

$$0 = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_i} \cdot \frac{\partial x_2}{\partial x_i} + \cdots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i}$$

This caused things to "go away"

For $i \neq k$ & $k = n$ we have that ~~the partials~~ $\frac{\partial x_k}{\partial x_i} = 0$

Thus we obtain:

$$0 = \frac{\partial F}{\partial x_i} \cdot \frac{\partial x_i}{\partial x_i} + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_f} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial N_n} \cdot \frac{\partial f}{\partial x_i}$$

Solving, we obtain $\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_n}$

$\boxed{///}$

Ex: Compute $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ for implicit function $z(x,y)$ given by

$$x^3 + y^3 + z^3 = 2xyz - 5$$

Sol: we want to use IFT. $x^3+y^3+z^3 = 2xyz - 5$ iff $x^3+y^3+z^3-2xyz+5=0$

Using $F(x,y,z) = x^3+y^3+z^3-2xyz+5$, we see

$\frac{\partial F}{\partial x} = 3x^2 - 2yz$, $\frac{\partial F}{\partial y} = 3y^2 - 2xz$, and $\frac{\partial F}{\partial z} = 3z^2 - 2xy$.

Hence, By IFT: $\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} = -\frac{3x^2 - 2yz}{3z^2 - 2xy}$ & $\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}$

$= -\frac{3y^2 - 2xz}{3z^2 - 2xy}$ $\boxed{\text{///}}$

What is the derivative of a multivariat function..?

Gradient AND Optimization

Goal: Optimize Functions of Several Variable By extending tricks From Calc I into multi variables.

Def: The gradient of a Function $f(x_1, x_2, \dots x_n)$ is

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots \frac{\partial f}{\partial x_n} \right\rangle$$

Note: gradient can Be used to clearly restate Many of the theorems "Some stuff" that we've seen

① Chain Rule: $\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}$

Why? $\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$

By the ChainRule

$$= \underbrace{\left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle}_{} \cdot \underbrace{\left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \dots, \frac{\partial x_n}{\partial t_i} \right\rangle}_{}$$

$$= \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}$$

Claim: Directional Derivatives can also Be expressed using the gradient...

Why?: Recall that the Directional Derivative of f at $\vec{P}$ in the direction of $\vec{u}$ is:

$$D_{\vec{u}} f(\vec{P}) = \lim_{h \to 0^+} \frac{f(\vec{P} + h\vec{u}) - f(\vec{P})}{h}$$

Define $g(h) = f(\vec{P} + h\vec{N})$ and Notice $g(0) = f(\vec{P})$

$\therefore D_{\vec{u}} f(\vec{P}) = \lim_{h \to 0^+} \frac{g(h) - g(0)}{N} = g'(0)$ on the other Hand,

$g'(h) = \frac{D}{Dh}[f(\vec{P} + h\vec{u})] = \frac{D}{Dh}[f(P_1 + hu_{(1)}), (P_2 + hu_2), \ldots, (P_N + hu_N)]$

Recognize this as a chain rule for $X_i = P_i + hu_i$: $\cancel{g'(h) = \frac{DF}{Dx}}$ Derive of

$g'(h) = \nabla f(\vec{P} + h\vec{u}) \cdot \frac{D\vec{X}}{Dh} = \nabla f(\vec{P} + h\vec{u}) \cdot \langle u_1, u_2, \ldots, u_n \rangle$

$= \nabla f(\vec{P} + hu) \cdot \vec{u}$ $\therefore$ we have $g'(0) = \nabla f(\vec{P} + 0u) \cdot \vec{u} = \nabla f(\vec{P}) \cdot \vec{u}$

Finally we see $D_{\vec{u}} f(\vec{P}) = \nabla f(\vec{P}) \cdot \vec{u}$

EX: lets compute the $D_{\vec{u}} f(\vec{P})$ for $f(x,y) = 4x\sqrt{y}$ at $\vec{P} = \langle 1,4 \rangle$ in Direction $\vec{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Sol; we Know $D_u f(\vec{P}) = \nabla f(\vec{P}) \cdot \vec{u}$
$\nabla f(x,y) = \langle 4y^{1/2}, 2xy^{-1/2} \rangle$. $\therefore \nabla f(\vec{P}) = \langle 4 \cdot 2, 2 \cdot 1 \cdot \frac{1}{2} \rangle = \langle 8, 1 \rangle$

$\therefore D_{\vec{u}} F(\vec{P}) = \nabla f(\vec{P}) \cdot \vec{u} = \langle 8, 1 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = -\frac{8}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$ ☒

5 Minute Break

Question; In which (Chris wants to do another gradient problem, we will return to this material later)

EX: Compute ∇f for $f(x,y,z) = \frac{xz}{y+z}$.

Sol; $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$. $\frac{\partial f}{\partial x} = \frac{z}{y+z}$, $\frac{\partial f}{\partial y} = -\frac{xz}{(y+z)^2}$, and ~~$\frac{\partial f}{\partial z} = (y+z)\frac{\partial}{\partial z}[xz]$ $-xz\frac{\partial}{\partial z}[y+z]$~~

$$\text{and } \frac{(y+z)\frac{\partial}{\partial z}[xy] - xz\frac{\partial}{\partial z}[y+z]}{(y+z)^2}$$

$$= \frac{(y+z)x - xz \cdot 1}{(y+z)^2}$$

$$= \frac{xy}{(y+z)}$$

$$\therefore = \left\langle \frac{z}{y+z}, \frac{-xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \right\rangle$$

Ex: How do we optimize the directional derivative?

Think about f at P an vary unit vector $\vec{u}$

$$D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = |\nabla f(\vec{P})||\vec{u}| \cos(\theta)$$

Just computing earlier

geo int of dot product

$= |\nabla f(\vec{P})| \cos(\theta)$ $\therefore$ Maximizing $D_u f(P)$ amounts to maximizing $\cos(\theta)$

$\vec{u}$ unit vector

We know from calc I $\cos(\theta)$ is maximized at $\cos(0)=1$ (from 0 to π)

$\therefore$ ① the direction of the gradient maximizes directional derivative.

② the maximum directional derivative of f is $|\nabla f(\vec{P})|$

EX: ~~[illegible]~~ Compute the direction of max value of $D_{\vec{u}} f(\vec{p})$

For $f(x, y, z) = \frac{xz}{y+z}$ at $\vec{p} = \langle 1, 1, -2 \rangle$

Sol: We already computed $\nabla f = \langle \frac{z}{y+z}, \frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \rangle$

$\therefore$ at $\vec{p} = \langle 1, 1, -2 \rangle$, the Dir. Derivative is maximized in direction

$\nabla f(1, 1, -2) = \langle \frac{-2}{1-2}, \frac{1(-2)}{(1-2)^2}, \frac{1 \cdot 1}{(1-2)^2} \rangle = \langle 2, 2, 1 \rangle$

Furthermore, the max value is $|\nabla f(\vec{p})| = |\langle 2, 2, 1 \rangle| = \sqrt{4+4+1} = 3$ $\boxed{\checkmark}$

chris says → (Read notes from Calc I about optimization) "You will need a very good grasp on optimization" (For tomorrow's class)

Def: A function f has...

① a local maximum value at $\vec{p}$ when $f(\vec{p}) \geq f(\vec{x})$ for all $\vec{x}$ near $\vec{p}$.

② a global maximum ~~point~~ value at $\vec{p}$ when $f(\vec{p}) \geq f(\vec{x})$ for all $\vec{x} \in \text{Dom}(f)$

(we call p the (local/global) maximum point for f).

③ minima (Both local & global) are defined similarly [Just flip inequalities

Recall: $f(x) = x$ has none of these...

Q: How do we guarantee existence of extrema? (maxima or extrema)

↳ where do we look for them?

Def: The critical points, point p, $\in \text{Dom}(f)$ a critical point of f when either $\nabla f(\vec{p})$ does not exist or $\nabla f(\vec{p}) = \vec{0}$

Prop (~~[illegible]~~ Fermat's extrema theorem): The local extrema of function f occur only at critical points of f